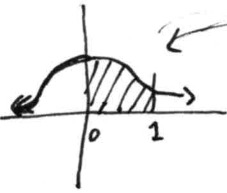
Name: Solutions Section: ______

Net Area and Integrals

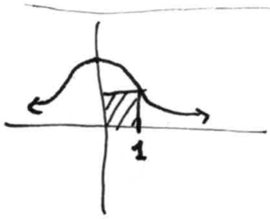


1. The integral $\int_a^b e^{-x^2}\,dx$ is important in statistics,[1] but it is infamously hard to compute. Many statistics textbooks include a table which lists the value of the integral for different values of a and b. We will use Reimann Sums to generate one of these approximations.

   (a) Express the integral $\int_0^1 e^{-x^2}\,dx$ as the limit of its Right Reimann Sums.

   $$\int_0^1 e^{(-x^2)}\,dx = \lim_{n\to\infty}\Big[\underbrace{\sum_{i=1}^{n} e^{-(x_i)^2}\cdot\Delta x}_{\text{area approximated using } n \text{ rectangles}}\Big]$$

   where $\boxed{\Delta x = \frac{1-0}{n} = \frac{1}{n}}$ and $x_i = a + i\cdot\Delta x$ $\boxed{x_i = \frac{i}{n}}$

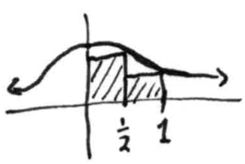
   

   (b) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 1$. Use a calculator to simplify.

   $R_1 = f(x_1)\Delta x$ $\quad \Delta x = \frac{1-0}{1} = 1 \quad x_1 = 1$

   $= e^{-(1)^2}\cdot 1 \approx 0.36788$

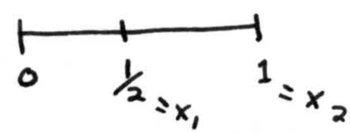
   

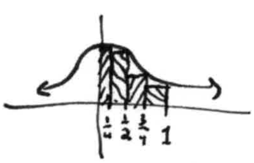
   

   (c) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 2$. Use a calculator to simplify.

   $R_2 = f(x_1)\Delta x + f(x_2)\Delta x \quad \Delta x = \frac{1-0}{2} = \frac{1}{2}$

   $= e^{-(.5)^2}(.5) + e^{-(1)^2}(.5) \approx 0.57334$

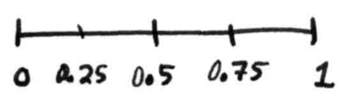
   

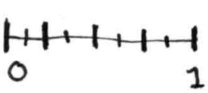
   (d) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 4$. Use a calculator to simplify.

   $R_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \quad \Delta x = \frac{1-0}{4} = 0.25$

   $= e^{-(.25)^2}(.25) + e^{-(.5)^2}(.25) + e^{-(.75)^2}(.25) + e^{-(1)^2}(.25)$

   ≈ 0.66397

   $\Delta x = \frac{1-0}{8} = .125$

   (e) Approximate $\int_0^1 e^{-x^2}\,dx$ using Right Sums and $n = 8$. Use a calculator to simplify.

   $R_8 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x$

   $= (.125)\left[e^{-(.125)^2} + e^{-(.25)^2} + e^{-(.375)^2} + e^{-(.5)^2} + e^{-(.625)^2} + e^{-(.75)^2} + e^{-(.875)^2} + e^{-(1)^2}\right]$

   ≈ 0.70636

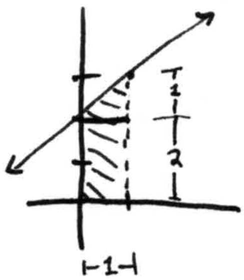
   (f) How do these compare to the correct value of $\int_0^1 e^{-x^2}\,dx = .7468241\ldots$?

   they get closer as n gets bigger!

[1] This and other similar integrals are needed to compute the probability of events that follow a normal distribution. See, for example, `http://en.wikipedia.org/wiki/Standard_normal_distribution#Cumulative_distribution`.

Name: ______________________ Section: ______________________

2. What is the graphical meaning of $\int_0^1 x + 2\,dx$? Compute this area geometrically.

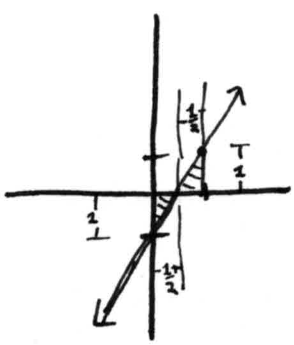


$$\int_0^1 (x+2)\,dx = \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 2\cdot 1$$

$$= \frac{1}{2} + 1$$

$$= 1.5$$

3. What is the graphical meaning of $\int_0^1 2x - 1\,dx$? Compute this area geometrically.



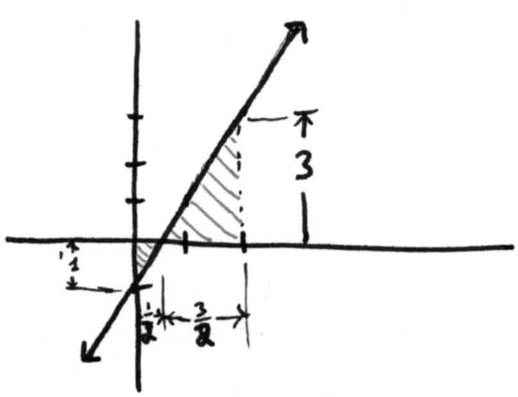
$$\int_0^1 [2x - 1]\,dx = (\text{area above}) - (\text{area below})$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot\frac{1}{2}\cdot 1$$

$$= \frac{1}{4} - \frac{1}{4}$$

$$= 0$$

4. What is the graphical meaning of $\int_0^2 2x - 1\,dx$? Compute this area geometrically.



$$\int_0^2 [2x-1]\,dx = (\text{area above}) - (\text{area below})$$

$$= \frac{1}{2}\cdot 3\cdot\frac{3}{2} - \frac{1}{2}\cdot 1\cdot\frac{1}{2}$$

$$= \frac{9}{4} - \frac{1}{4} = \frac{8}{4} = 2$$

Name: ______________________ Section: ______________________

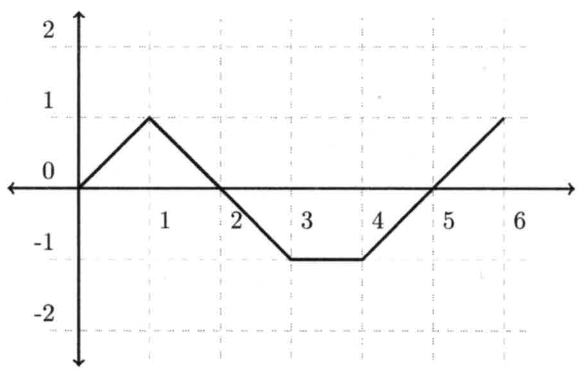
5. Answer the following **Yes or No**

   (a) Can you distribute integrals across integrals?
   i.e. is $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$? Yes No

   (b) Can you pull constants through integrals?
   i.e. is $\int_a^b (c \cdot f(x))\, dx = c \cdot \left(\int_a^b f(x)\, dx \right)$? Yes No

   (c) Can you distribute integrals across products? Yes No!

6. Suppose that the function $f(x)$ is given by the following graph.



Let $A(x) = \int_0^x f(t)\, dt$. Compute the following

(a) $A(1) =$ $\frac{1}{2}$

(b) $A(2) =$ $\frac{1}{2} + \frac{1}{2} = 1$

(c) $A(4) =$ $\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - 1 = -\frac{1}{2}$

Remember that $A'(x) = \frac{d}{dx}\left[\int_0^x f(t)\, dt\right] = f(x)$.

(d) $A'(1) =$ $f(1) = 1$

(e) $A'(2) =$ $f(2) = 0$

(f) $A'(4) =$ $f(4) = -1$

Integration Review Worksheet Math 1071 Class #27

Name: ____________________ Section: ____________________

Computing Integrals Quickly

1. $\int_{-1}^{1} \left[e^x - 4x + 6x^2\right]\, dx$

$$= \int_{-1}^{1} e^x\, dx - 4\int_{-1}^{1} x\, dx + 6\int_{-1}^{1} x^2\, dx$$

$$= \left[e^x - 4\cdot\frac{x^2}{2} + 6\cdot\frac{x^3}{3}\right]_{-1}^{1} = \left[e^x - 2x^2 + 2x^3\right]_{-1}^{1}$$

$$= \left[e^1 - 2(1)^2 + 2(1)^3\right] - \left[e^{-1} - 2(-1)^2 + 2(-1)^3\right]$$

$$= (e - 2 + 2) - (e^{-1} - 2 - 2) = e - e^{-1} - 4$$

---

2. $\int_{1}^{2} \frac{x^2 + x + 1}{x}\, dx$

$$= \int_{1}^{2} \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x}\, dx = \int_{1}^{2}\left[x + 1 + \frac{1}{x}\right] dx$$

$$= \left[\frac{x^2}{2} + x + \ln|x|\right]_{1}^{2}$$

$$= \left(\frac{4}{2} + 2 + \ln|2|\right) - \left(\frac{1}{2} + 1 + \ln|1|\right)$$

$$= 4 + \ln|2| - 1.5 + 0 = 2.5 + \ln(2)$$

---

3. $\int_{-1}^{1} (x^2 + 3)(x - 1)\, dx$

$$= \int_{-1}^{1} x^3 - x^2 + 3x - 3\, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} + \frac{3x^2}{2} - 3x\right]_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{3} + \frac{3}{2} - 3\right) - \left(\frac{\overset{1}{(-1)^4}}{4} \overbrace{- \frac{\overset{-1}{(-1)^3}}{3}}^{+\frac{1}{3}} + \frac{3\overset{1}{(-1)^2}}{2} \overbrace{- 3(-1)}^{+3}\right)$$

$$= \left(\frac{1}{4} - \frac{1}{3} + \frac{3}{2} - 3\right) - \left(\frac{1}{4} + \frac{1}{3} + \frac{3}{2} + 3\right)$$

$$= 0 + -\frac{2}{3} + 0 + -6 = -\frac{2}{3} - 6$$

Name: ______________________ Section: ______________________

4. $\int_0^3 x\sqrt{x^2+16}\,dx$ $\frac{du}{2}$

$$\left(\begin{array}{l|l} u = x^2+16 & x=0 \Rightarrow u = 0^2+16 = 16 \\ \frac{du}{dx} = 2x & x=3 \Rightarrow u = 3^2+16 = 25 \\ \frac{du}{2} = x\,dx & \end{array}\right)$$

$$= \int_{16}^{25} (u)^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\left[\frac{u^{3/2}}{3/2}\right]_{16}^{25} = \frac{1}{2}\left[\frac{(\sqrt{25})^3}{3/2} - \frac{(\sqrt{16})^3}{\frac{3}{2}}\right] = \frac{5^3}{3} - \frac{4^3}{3} = \frac{61}{3}$$

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5. $\int_0^5 \frac{1}{x+1}\,dx$

$$\left(\begin{array}{l|l} u = x+1 & x=0 \Rightarrow u = 1 \\ \frac{du}{dx} = 1 & x=5 \Rightarrow u = 6 \\ du = dx & \end{array}\right)$$

$$= \int_1^6 \frac{1}{u}\,du = \left[\ln|u|\right]_1^6 = \ln(6) - \ln(1)$$

$$= \ln(6)$$

---

6. $\int_0^1 x\,e^{x^2+1}\,dx$ $\frac{du}{2}$

$$\left(\begin{array}{l|l} u = x^2+1 & x=0 \Rightarrow u = 0^2+1 = 1 \\ \frac{du}{dx} = 2x & x=1 \Rightarrow u = 1^2+1 = 2 \\ \frac{du}{2} = x\,dx & \end{array}\right)$$

$$= \int_1^2 e^u \cdot \frac{du}{2} = \left[\frac{1}{2}e^u\right]_1^2 = \frac{1}{2}\left[e^2 - e^1\right]$$

Name: ______________________ Section: ______________________

7. Compute $\int_1^2 \frac{x^2}{\boxed{x^3+1}} \boxed{dx}$ $\frac{du}{3x^2}$

$u = x^3 + 1$

$\frac{du}{dx} = 3 \cdot x^2$

$du = 3x^2 \cdot dx$

$dx = \frac{du}{3x^2}$

$x = 1 \Rightarrow u = 1^3 + 1 = 2$

$x = 2 \Rightarrow u = 2^3 + 1 = 9$

$$= \int_2^9 \frac{\cancel{x^2}}{u} \cdot \frac{du}{3\cancel{x^2}} = \frac{1}{3} \cdot \int_2^9 \frac{1}{u}\, du = \left[\frac{1}{3} \cdot \ln(u)\right]_2^9$$

$$= \frac{1}{3} \cdot \ln(9) - \frac{1}{3} \cdot \ln(2)$$

8. Compute $\int_1^2 \frac{x^3+1}{x^2}\, dx$

<u>Cannot</u> do u-sub.
<u>Must</u> rewrite.

$$= \int_1^2 \frac{x^3}{x^2} + \frac{1}{x^2}\, dx$$

$$= \int_1^2 x + x^{-2}\, dx$$

$$= \left[\frac{x^2}{2} + \frac{x^{-1}}{-1}\right]_1^2$$

$$= \left[\frac{x^2}{2} - \frac{1}{x}\right]_1^2 = \left(\frac{4}{2} - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{1}\right) = 2$$

Name: ____________________ Section: ____________________

9. Compute $\int \left[\frac{x}{x^2+4} + \frac{x^2+4}{x} \right] dx$

must split into TWO parts

$$= \int \frac{x}{\boxed{x^2+4}} dx + \int \frac{x^2+4}{x} dx$$

$u = x^2+4$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$$= \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}} + \int \left[\frac{x^2}{x} + \frac{4}{x}\right] dx$$

$$= \frac{1}{2}\int \frac{1}{u} du + \int x\, dx + 4 \cdot \int \frac{1}{x} dx$$

$$= \frac{1}{2} \cdot \ln|u| + \frac{x^2}{2} + 4 \cdot \ln|x| + C$$

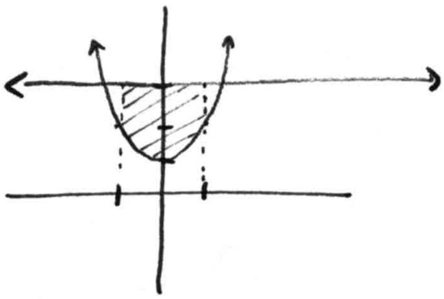
$$= \frac{1}{2} \cdot \ln|x^2+4| + \frac{x^2}{2} + 4 \cdot \ln|x| + C$$

Integration Review Worksheet Math 1071 Class #27

Name: ____________________ Section: ____________________

The Area Between Curves

1. Find the area between $y = x^2 + 1$ and $y = 3$ on $[-1, 1]$.



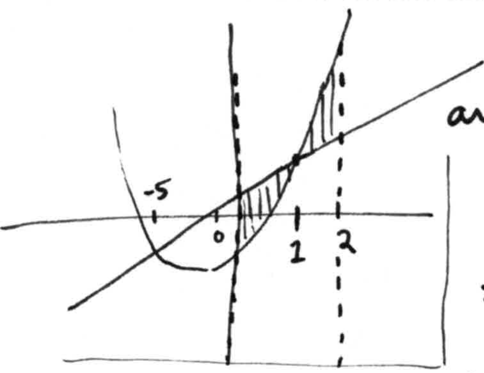
$$\int_{-1}^{1}[3-(x^2+1)]\,dx$$

$$=\int_{-1}^{1}[2-x^2]\,dx$$

$$=\left[2x-\frac{x^3}{3}\right]_{-1}^{1} = \left(2\cdot1-\frac{(1)^3}{3}\right)-\left(2(-1)-\frac{(-1)^3}{3}\right)=\left(2-\frac{1}{3}\right)-\left(-2-\frac{1}{3}\right)$$

$$=4-\frac{2}{3}$$

2. Find the area between $y = x^2 + 6x - 2$ and $y = 2x + 3$ on $[0, 2]$.



$$\text{area}=\int_0^1\left[(2x+3)-(x^2+6x-2)\right]dx + \int_1^2 (x^2+6x-2)-(2x+3)\,dx$$

$$=\int_0^1[-x^2-4x+5]\,dx+\int_1^2[x^2+4x-5]\,dx$$

Intersect when

$x^2+6x-2 = 2x+3$

$x^2+4x-5=0$

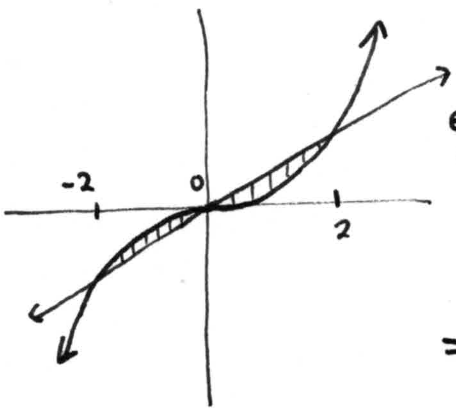
$(x+5)(x-1)=0$

when $x=-5$ & $x=1$

$$=\left((-1-4+5)-(0-0+5)\right)+\left((4+4-5)-(1+4-5)\right)$$

$$=(0-5)+(3-0)=-2$$

3. Find the area enclosed by $y = x^3$ and $y = 4x$.



$$\text{enclosed area}=\int_{-2}^{0}[x^3-4x]\,dx + \int_0^2[4x-x^3]\,dx$$

Intersect when

$x^3=4x$

when $x=0$ when $x^2=4$

$x=\pm 2$

$$=\left[\frac{x^4}{4}-2x^2\right]_{-2}^{0}+\left[2x^2-\frac{x^4}{4}\right]_0^2$$

$$=(0-0)-\left(\frac{(-2)^2}{4}-2(-2)^2\right)+\left(2\cdot2^2-\frac{2^4}{4}\right)-(0-0)$$

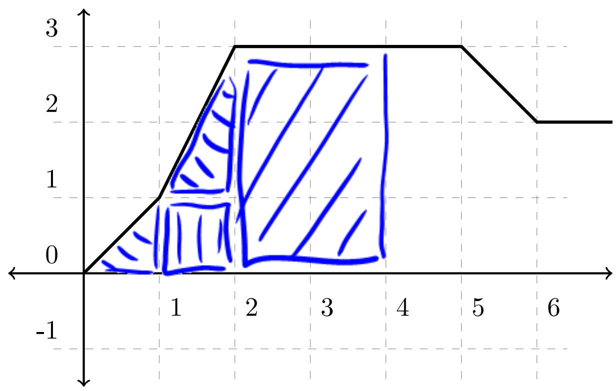
$$=0-(1-8)+{}^{6}(8-1)-0$$

$$=7+7=14 \Leftarrow$$

Name: ______________________ Section: ______________________

Applications to Business Concepts

1. Suppose that the ***marginal*** cost $C'(x)$ function is graphed below.



Suppose also that that $C(0) = 2$.

(a) Compute the total cost of selling 2 units.

Know $$\underbrace{\int_0^2 C'(x)\,dx}_{\text{given}} = \underbrace{C(2)}_{\text{want}} - \underbrace{C(0)}_{\text{given}}$$

geometrically $$\int_0^2 C'(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 1\cdot 2 + 1\cdot 1$$
$$= 2.5$$

so $2.5 = C(2) - 2$

so $\boxed{C(2) = 4.5}$

(b) Compute the total cost of selling 4 units.

Know $$\int_0^4 C'(x)\,dx = C(4) - C(0)$$

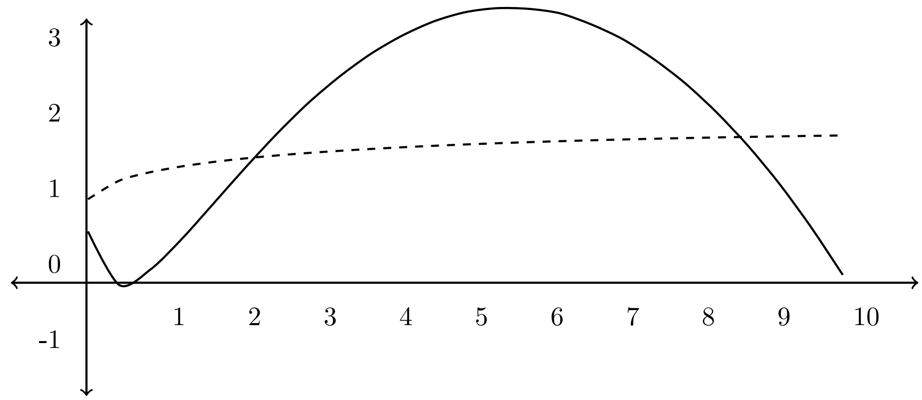
geometrically $$\int_0^4 C'(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot 2\cdot 1 + 1\cdot 1 + 3\cdot 2$$
$$= 8.5$$

So $8.5 = C(4) - 2$

$C(4) = 10.5$

Name: ____________________ Section: ____________________

2. Suppose that the ***marginal*** cost $C'(x)$ (dashed) and the ***marginal*** revenue $R'(x)$ (solid) functions are given by the following graphs.



(a) Suppose that $\int_0^6 C'(x)\,dx = 10$ and that $\int_0^6 R'(x)\,dx = 20$. Suppose also that $C(0) = 5$ and that $R(0) = 0$. Approximate the total profit from selling 6 units.

Know $P(x) = R(x) - C(x)$ $\Rightarrow P(0) = 0 - 5 = -5$

$P'(x) = R'(x) - C'(x)$

And $\int_0^6 P'(x) = P(6) - P(0)$

So $\int_0^6 R'(x) - C'(x)\,dx = P(6) - (-5)$

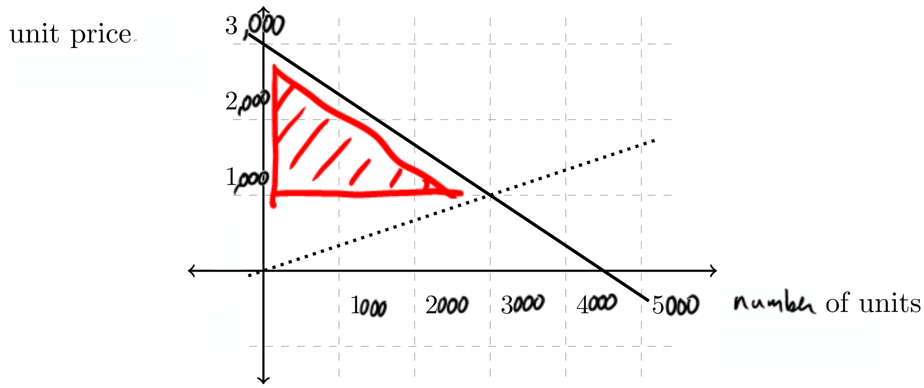
So $\int_0^6 R'(x)\,dx - \int_0^6 C'(x)\,dx = P(6) + 5$

So $\underbrace{20 - 10}_{10} = P(6) + 5$

$P(6) = 5$

Name: ______________________________ Section: ______________________________

3. Suppose that the supply price curve (dotted) and the demand price curve (solid) functions are given by the following graphs.



(a) Use the graph to estimate the equilibrium price and quantity.

equilibrium quantity = 3,000 units

equilibrium price = 1,000 \$/unit

(b) Compute the consumer surplus.

consumer surplus $= \frac{1}{2} \cdot 3000 \cdot 2000$

$= 3{,}000{,}000$ \$

(c) (Supplemental). Use the limit definition of the integral to explain why the consumer surplus as computed in (b) is an honest approximation of

> "The difference between the maximum amount consumers would be willing to pay and the amount they actually pay for a good"[2]

Area computed above $= \int_0^3 P(x) - 1000 \, dx$

$$= \lim_{n \to \infty} \left[\sum_{i=1}^{n} (P(x_i) - 1000) \Delta x \right]$$

$$\approx \underbrace{\sum_{i=1}^{3000} \left[P(i) - 1000 \right] \cdot 1}$$

If $n = 3000$

$$\Delta x = \frac{3000 - 0}{3000} = 1$$

$$x_i = i$$

this is what you get when you add up, for each i, the amount beyond 1000 that **someone** would have paid for **just the ith unit**

[2] *Microeconomics: Private & Public Choice*, Gwartney et. al.